J. Fluid Mech. (1988), *vol.* 192, *pp.* 329–337
Printed in Great Britain

Cyclic recurrence in nonlinear unidirectional ocean waves

By PETER J. BRYANT
Mathematics Department, University of Canterbury, Christchurch, New Zealand

(Received 8 July 1987 and in revised form 4 November 1987)

A fully nonlinear model is developed for the unidirectional propagation of periodic gravity wave groups in deep water, in which the shape of the group envelopes changes cyclically. It is intended to describe the slow-time evolution of wave groups on the open ocean surface, and to generalize the cyclic recurrence that can occur during the sideband modulation of Stokes waves and Schrödinger wave groups. The weak nonlinear interactions are shown to concentrate the wave energy at the centre of each group at regular intervals, causing the waves there to be of greater height locally in space and time. This is suggested as one mechanism for the local wave breaking that is observed on the open ocean surface. The cyclically recurring wave groups may be interpreted as the limit-cycle stage in a progression from uniform wave groups to chaos on the forced, damped, ocean surface.

1. Introduction

The open ocean surface often has the appearance of an ensemble of gravity wave groups, with each group being identifiable over a number of wavelengths along and transverse to the direction of wave propagation. This is consistent with the common belief about the regular arrival of waves larger than the others, and may be associated with the presence of underlying swell. The linear theory predicts that a group should slowly disintegrate as the wave components composing it disperse. The weakly nonlinear theory, modelled by the nonlinear Schrödinger equation, shows that groups can have constant envelopes (envelope solitons) when the linear dispersion is balanced by weak nonlinear interactions. Analysis of this equation (§3) shows also that it has group solutions whose envelope shape recurs cyclically. It can be expected that gravity wave groups on the ocean surface do not have envelopes of constant shape (ignoring forcing and damping), but rather that dispersive and nonlinear effects are not in balance, causing the envelopes to change slowly in time. The particular case examined here, in which the slow change in shape of the envelopes is cyclic, is intended as a model for more general shape evolution in time.

Another reason for investigating cyclic recurrence is its role in the long-time evolution of gravity wavetrains on deep water. Lake *et al.* (1977) showed experimentally, and by computation from the nonlinear Schrödinger equation, that a modulated wavetrain may exhibit recurrence by returning cyclically near to its initial condition. Yuen & Ferguson (1978) continued this investigation with a more detailed analysis of the dependence of the computed wavetrains on the initial conditions, finding and explaining how the pattern of recurrence could be simple or complex. Bryant (1982) found that much smaller changes occur during the modulation cycle for initial wave groups of constant envelope than for initial uniform waves of constant shape. The interpretation is that the modulated wave groups are almost in balance between dispersive and nonlinear effects, but the modulated waves are much further from this balance. That investigation was based on a numerical solution of the weakly nonlinear evolution equations with the quadratic and resonant tertiary interactions retained. Reservations about weakly nonlinear approximations led to the development of the present fully nonlinear method. Dold & Peregrine (1986) use a boundary-integral method for solving the two-dimensional nonlinear unsteady water wave problem to calculate the evolution of an initial modulated wavetrain. Some of their examples demonstrate cyclic recurrence, and some show wave breaking.

The common feature of these previous investigations is the numerical time-stepping of evolution equations for nonlinear water waves, beginning with an initial modulated wavetrain. Cyclic recurrence is found to occur, in the sense that the wavetrain returns cyclically near to the initial state. Because this property is important to ocean waves, it seems desirable to calculate directly the exact cyclic recurrent wave solutions of the nonlinear evolution equations, as a check on the time-stepped solutions, and to understand better the nonlinear structure of ocean waves. The aim here is to investigate general forms of fully nonlinear unidirectional wave groups with cyclic recurrence, for which the modulated wavetrains are a particular case.

The numerical method is a generalization of that developed previously (Bryant 1985) for nonlinear waves in deep water. The unidirectional cyclic recurrent wave groups are represented by a triple Fourier series in space, fast time, and slow time, whose coefficients are chosen so that the water-surface displacement and velocity potential satisfy the fully nonlinear water-surface boundary conditions over a network of points in space and time. The terms in the Fourier series are those that contribute above a small numerical level of significance to the nonlinear boundary conditions. Although it had been intended to calculate the linear stability and time evolution of perturbed recurrent wave groups, the number of wave components needed for their description proved to be too large to make these computations accurately at the present time. Empirical relations for forcing and damping, both with wave number or frequency dependence, could have been added to the model without difficulty, but it was decided as a first step to confine attention to the dispersive and fully nonlinear contributions only.

2. Wave-group description

A slowly varying wavetrain of typical wavelength $2\pi l$, typical wave amplitude a, and fixed group length $2\pi L$, propagates on deep water in the x-direction. The x- and y-variables are non-dimensional multiples of l, t is a multiple of $(l/g)^{\frac{1}{2}}$, the water-surface displacement $\eta(x,t)$ is a multiple of a, and the velocity potential $\phi(x,y,t)$ is a multiple of $(gl)^{\frac{1}{2}}a$. The ratio $\epsilon = a/l$ is a measure of wave slope. The non-dimensional governing equations are

$$\phi_{xx}+\phi_{yy} = 0, \quad y < \epsilon\eta(x,t), \tag{2.1a}$$

$$\phi_x, \phi_y \rightarrow 0, \quad y \rightarrow -\infty, \tag{2.1b}$$

$$\eta_t - \phi_y + \epsilon\eta_x\phi_x = 0, \quad y = \epsilon\eta(x,t), \tag{2.1c}$$

$$\eta + \phi_t + \tfrac{1}{2}\epsilon(\phi_x^2+\phi_y^2) = 0, \quad y = \epsilon\eta(x,t). \tag{2.1d}$$

The wavenumbers k are non-dimensional integer multiples of $1/L$, with the wavenumbers in the fundamental waveband centred on $k_0 = L/l$ (which need not be an integer). The wave phase in the fundamental waveband may be written

$$\frac{k}{k_0}x-\omega t = \frac{k}{k_0}x-\left(1+\frac{k-k_0}{2k_0}+O\left(\frac{k-k_0}{k_0}\right)^2\right)t, \tag{2.2}$$

for waves on deep water. Nonlinear wave interactions produce harmonic wavebands centred on jk_0, $j = 0, 1, 2, \ldots$. The waveband $j = 0$ describes the set-down and mean velocity associated with each wave group, and $j = 1$ is the fundamental waveband. When the phase in (2.2) is generalized to other wavebands, the water-surface displacement may be written

$$\eta = \sum_{j=0}^{J}\sum_{k=k_1(j)}^{k_2(j)} a_{jk}\cos\left(\frac{k}{k_0}(x-\tfrac{1}{2}t)-j(\tfrac{1}{2}+\beta)t\right). \tag{2.3}$$

The nonlinear frequency correction β is $O((k-k_0)/k_0)^2$ provided the dispersive and nonlinear effects are near balance. If the coefficients a_{jk} are all constant, this double Fourier series describes periodic wave groups of constant envelope. It should be appreciated that a Fourier series expansion for nonlinear wave groups of constant envelope must be a double series such as (2.3), with one index (j) numbering the different wavebands, and the other index (k) numbering the wave components within each waveband. Particular values of the index k occur in more than one j waveband.

Wave groups with cyclic recurrence are described by (2.3) if the coefficients a_{jk} have a slow periodic dependence on t. This periodic dependence can be included with the wave phase, when the water-surface displacement is written

$$\eta = \sum_{j_0=0}^{J}\sum_{j_1=J_1(j_0)}^{J_2(j_0)}\sum_{k=k_1(j_0,j_1)}^{k_2(j_0,j_1)} a_{j_0j_1k}\cos\left(\frac{k}{k_0}(x-\tfrac{1}{2}t)-\tfrac{1}{2}j_0t-j_1\alpha t\right). \tag{2.4a}$$

The associated velocity potential is

$$\phi = \sum_{j_0=0}^{J}\sum_{j_1=J_1(j_0)}^{J_2(j_0)}\sum_{k=k_1(j_0,j_1)}^{k_2(j_0,j_1)} b_{j_0j_1k}\exp\frac{|k|y}{k_0}\sin\left(\frac{k}{k_0}(x-\tfrac{1}{2}t)-\tfrac{1}{2}j_0t-j_1\alpha t\right). \tag{2.4b}$$

The frequency of cyclic recurrence α is much less than 1, and the $j\beta t$ contribution in (2.3) has been absorbed into the $j_1\alpha t$ contribution. In a frame of reference moving with the group, the term $\tfrac{1}{2}j_0t$ describes the fast-time variation of the wavetrain relative to the group, while the term $j_1\alpha t$ describes the slow-time variation of the shape of the group envelope. If the Fourier coefficients $a_{j_0j_1k}, b_{j_0j_1k}$ in (2.4) are all constant, these triple Fourier series describe wave groups with cyclic recurrence. Conversely, Fourier series expansions for nonlinear wave groups with cyclic recurrence must be triple series such as (2.4) if the Fourier coefficients are all constant. Particular values of the index k occur in more than one j_0,j_1 waveband.

As the wave slope ϵ is increased, the centre of the calculated fundamental waveband increases from the given value of k_0. Rescaling k_0 to the calculated value is equivalent to increasing the group propagation velocity from the linear value $\tfrac{1}{2}$ to the appropriate nonlinear value. There is no loss of generality in retaining $\tfrac{1}{2}$ as the group velocity in (2.4a,b), provided this rescaling is done before assigning a correct group velocity to any calculated example.

It should be emphasized that (2.4a,b) describe particular unidirectional periodic wave groups on deep water with cyclic recurrence. A general form for such groups must allow for more general phase relationships between the Fourier wave components. Equations (2.4a, b) can be made to satisfy (2.1) to within a prescribed small numerical precision, and it is this that is taken as the justification for studying (2.4a, b) as a model of ocean wave groups.

3. Nonlinear Schrödinger equation

The slow evolution of a weakly nonlinear unidirectional wavetrain on deep water is modelled by the nonlinear Schrödinger equation, which in the present non-dimensional notation is

$$\mathrm{i}(A_t+\tfrac{1}{2}A_x)-\tfrac{1}{8}A_{xx}-\tfrac{1}{2}\epsilon^2|A|^2A=0, \tag{3.1a}$$

where

$$\eta=R\{A(x,t)\exp\mathrm{i}(x-t)\}. \tag{3.1b}$$

The surface displacement is approximated by the fundamental waveband alone, when (2.4a) reduces to

$$\eta=\sum_{j=J_1}^{J_2}\sum_{k=k_1(j)}^{k_2(j)} a_{1jk}\cos\left(x-t+\frac{k-k_0}{k_0}(x-\tfrac{1}{2}t)-j\alpha t\right). \tag{3.2}$$

Cyclic recurrent solutions of the nonlinear Schrödinger equation are sought for which

$$A(x,t)=\sum_{j=J_1}^{J_2}\sum_{k=k_1(j)}^{k_2(j)} a_{1jk}\exp\mathrm{i}\left(\frac{k-k_0}{k_0}(x-\tfrac{1}{2}t)-j\alpha t\right). \tag{3.3}$$

If k_0 is an integer and the amplitudes are symmetric about $k=k_0$, then

$$A(x,t)=\sum_{j=J_1}^{J_2}\sum_{k=0}^{K(j)} A_{jk}\cos\frac{k}{k_0}(x-\tfrac{1}{2}t)\exp(-\mathrm{i}j\alpha t), \tag{3.4a}$$

where $k-k_0$ is replaced by k, and

$$A_{j0}=a_{1jk_0},\quad A_{jk}=2a_{1jk_0\pm k}\quad (k\neq 0). \tag{3.4b}$$

Equation (3.4a) may be rewritten

$$A(x,t)=\sum_{j=J_1}^{J_2} F_j(x-\tfrac{1}{2}t)\exp(-\mathrm{i}j\alpha t), \tag{3.5}$$

where each F_j is symmetric.

The known envelope solutions of constant shape may be obtained analytically by reducing (3.5) to the single $j=1$ term, substituting in (3.1a), and solving for F_1. The factor $\exp(-\mathrm{i}\alpha t)$ then describes a small frequency correction to the carrier wave. The analytical solution can be extended to simple cyclic recurrent wave groups, for example by reducing (3.5) to three terms, $j=0,1,2$, and substituting in (3.1a). The small quantity α is then the cyclic frequency for the shape evolution of the group envelopes.

There is no difficulty in calculating numerically the cyclic recurrent wave groups of the form of (3.4), (3.5). This is done by substituting (3.4a) into (3.1a) using trial values for the Fourier coefficients A_{jk} and the frequency α. The trial values are then improved by Newton's method so that (3.4a) satisfies (3.1a) and the normalization constraint (3.6) to any prescribed small numerical precision.

The wave-slope parameter ϵ is defined from the amplitude at the centre of each group at time $t=0$, equivalent to the normalization

$$\sum_j\sum_k A_{jk}=1. \tag{3.6}$$

Cyclic recurrent wave-group solutions of the form of (3.4*a*) could be found only for values of ϵ greater than about 0.2. Although these are solutions of the nonlinear Schrödinger equation (3.1), they are not satisfactory solutions for nonlinear waves in deep water because the weakly nonlinear assumption has doubtful validity at these large values of ϵ. However, they do provide satisfactory first estimates to the fully nonlinear, multi-waveband, cyclic recurrent wave-group solutions, with the conversion stated in (3.4*b*). The calculated Schrödinger solutions are used for this purpose only, and have not been investigated otherwise.

The cyclic recurrent solutions of the nonlinear Schrödinger equation show reasonable qualitative agreement with the corresponding fully nonlinear solutions, but the quantitative agreement is poor. In particular, the frequency of cyclic recurrence, α, can differ by a factor of two or more between the two solutions at the same values of ϵ and k_0. This wide difference for α is to be expected because cyclic resonance is a nonlinear phenomenon, and the Schrödinger equation provides only a first approximation to the nonlinear wave interactions. For the same reason, the Schrödinger solutions underestimate the steepening and growth of the central waves in each wave group during the cycle of recurrence. Substitution of the Schrödinger solutions with their associated velocity potentials into the nonlinear boundary conditions (2.1*c*, *d*) reveals large residuals at the higher frequencies and wave-numbers.

4. Fully nonlinear groups

The triple Fourier series expansions for η and ϕ (equations 2.4*a*, *b*) are substituted into the nonlinear boundary conditions (equations 2.1*c*, *d*) with trial values for the Fourier coefficients. The boundary conditions are then expressed also as triple Fourier series expansions. The trial values are improved by Newton's method until all Fourier coefficients in the expansions of the boundary conditions are less than some small numerical error. This is equivalent to satisfying the boundary conditions to within a small numerical error over a three-dimensional network of points in $x-\frac{1}{2}t$, t, and αt. The normalization constraint at the origin in physical space, the triple series equivalent of (3.6), is imposed also.

The procedure adopted is to begin with a particular single-waveband solution of the nonlinear Schrödinger equation (3.4), then to add wave components and wavebands to (2.4*a*, *b*) until no more are required, for a given numerical precision. Although adequate precision was achieved for the single-waveband Schrödinger solutions with only 20–30 wave components, many more wave components are needed for the full solutions. To conserve computer resources, no more than about 900 wave components were used, and these were placed in the wavebands to achieve the best numerical precision. The large number of wave components results from the steepness of the waves during the cycle of recurrence. This steepness is not modelled adequately by the single-waveband Schrödinger solutions, and requires the inclusion of higher wavebands to achieve satisfactory precision.

A number of families of cyclic wave-group solutions have been found and investigated. The dependence of the cyclic frequency α on the amplitude ratio ϵ for one such family when $k_0 = 5$ is sketched in figure 1. Each solution occurs at two points on this closed curve, once with the value of ϵ appropriate to $\alpha t = 0$, and again with the value of ϵ appropriate to $\alpha t = \pi$. The solution at the point A, for instance, where $\epsilon = 0.160$, $\alpha = 0.01357$, occurs again at the point B where $\epsilon = 0.262$. This means that the wave slope at the centre of the group in this example increases from

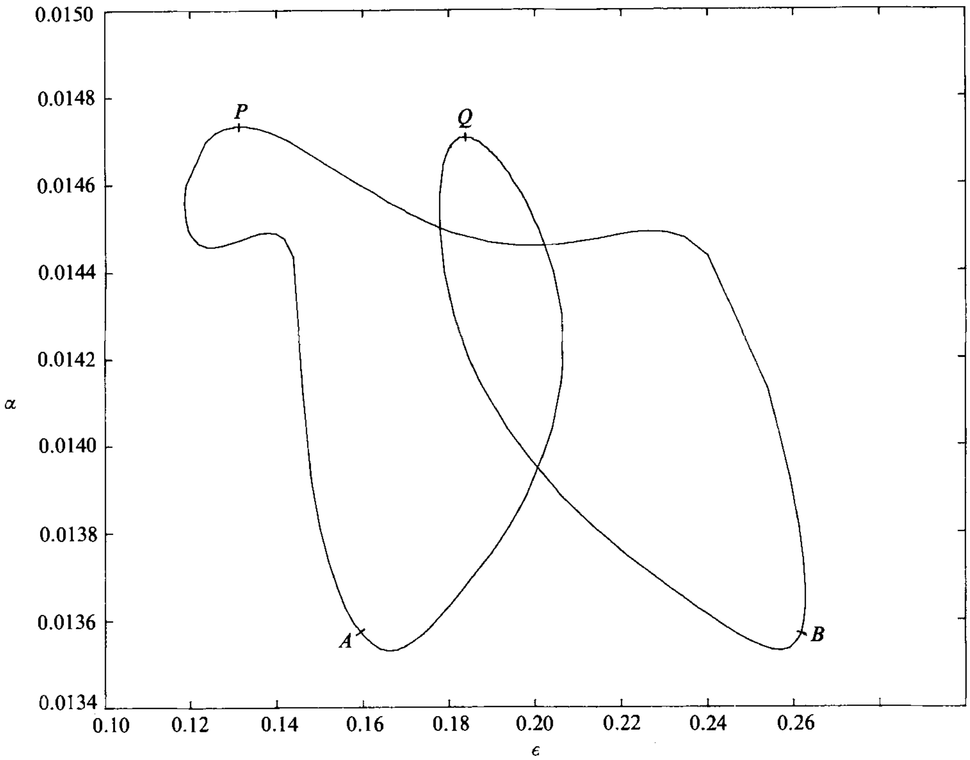

FIGURE 1. The cyclic frequency α *vs.* the amplitude parameter ϵ for one family of cyclic wave-group solutions.

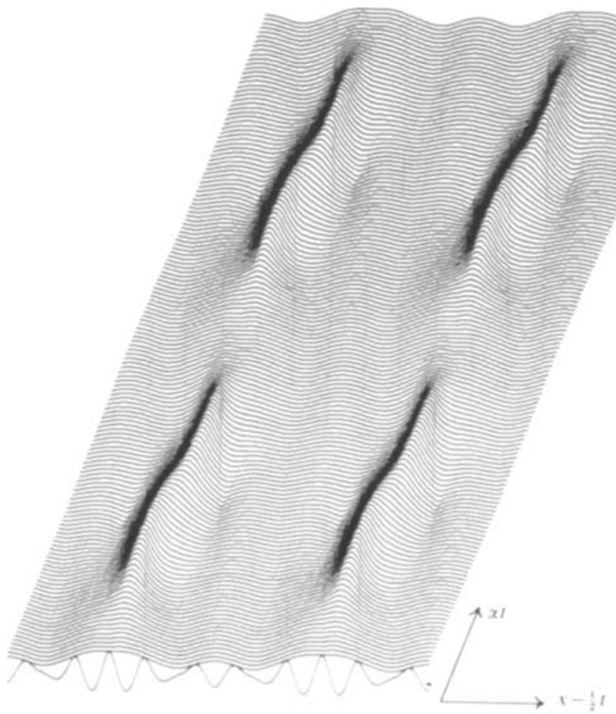

FIGURE 2. The time evolution of the envelope of the first cyclic wave-group example, including the initial wavetrain. Vertical magnification 20.

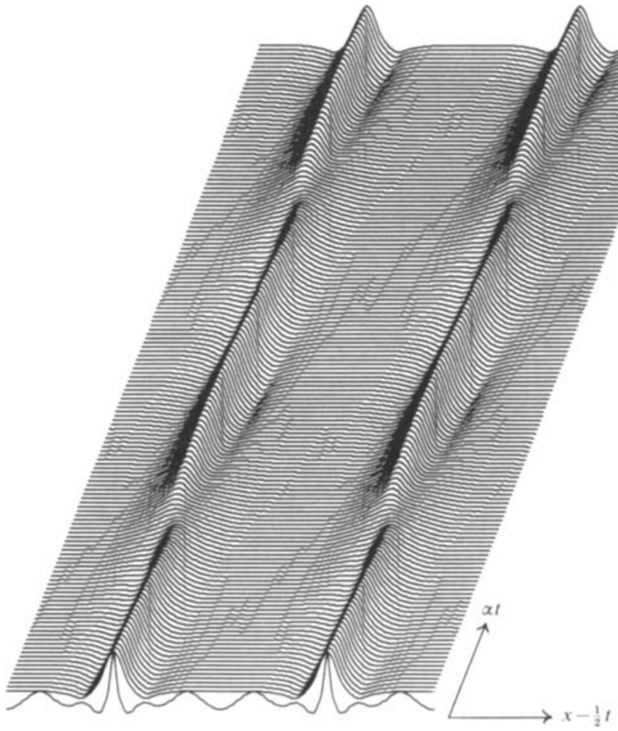

FIGURE 3. The time evolution of the envelope of the second cyclic wave-group example, including the initial wavetrain. Vertical magnification 20.

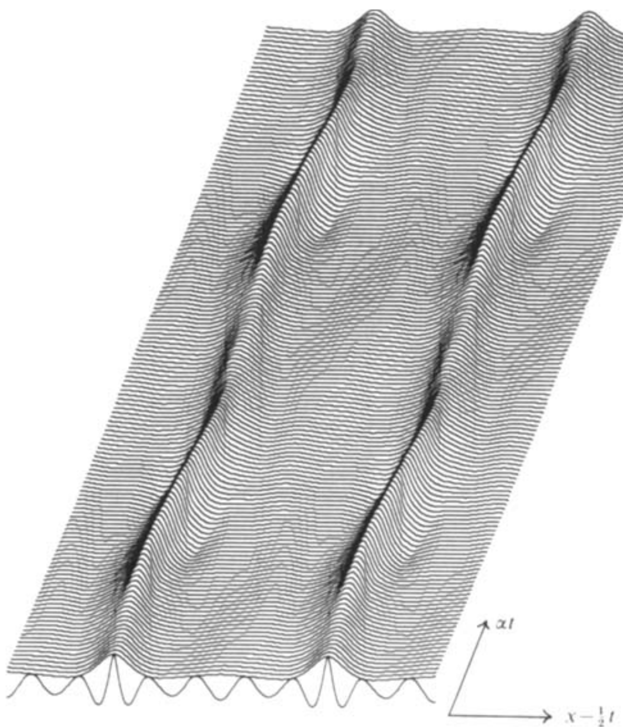

FIGURE 4. The time evolution of the envelope of the third cyclic wave-group example, including the initial wavetrain. Vertical magnification 20.

0.160 to 0.262 over half the cycle of recurrence. The time evolution of the envelope for this solution is illustrated in figure 2, with the initial wavetrain sketched also. Two group lengths relative to a frame of reference moving with the group velocity are shown across the figure. The slow-time evolution over two cycles or recurrence is drawn in perspective up the figure. It can be seen in the figure how each group sharpens and steepens towards the centre of each cycle, as the energy of the group becomes concentrated at its centre.

The points P and Q on the curve of figure 1 are the two points where the initial envelope and the envelopes at $\alpha t = \pi$ are the same. They are both points of bifurcation to new families of solutions for which the cyclic period is halved. The wave components in the new families of solutions are non-zero in (2.4a, b) only for the odd values of j_1 in the odd wavebands and the even values of j_1 in the even wavebands. The time evolution of the envelope for an example in the bifurcated family is sketched in figure 3. The two cycles of recurrence up the figure in this example extend from $\alpha t = 0$ to $\alpha t = 2\pi$, rather than from $\alpha t = 0$ to $\alpha t = 4\pi$ as in figure 2, because the double increment of j_1 in each waveband causes the envelope to have a period π/α. The initial values of ϵ is 0.360, decreasing to 0.208 at $\alpha t = \frac{1}{2}\pi$, with $\alpha = 0.021\,54$. The maximum value of ϵ for Stokes waves of steady shape is about 0.44, suggesting that a small increase in the minimum value of ϵ in this example could cause the maximum value of ϵ during the cycle of recurrence to be sufficiently large for wave breaking to occur. The wave component having the largest magnitude occurs in the fundamental waveband at $k = 3$, followed by those at $k = 4$, $5(= k_0)$, and 6. This is consistent with the initial wave profile sketched in figure 3, which shows three dominant waves per group length.

The third example, sketched in figure 4 from one of the families with all values of j_1 present in each j_0 waveband, rises from $\epsilon = 0.240$ initially to $\epsilon = 0.362$ at $\alpha t = \pi$, with $\alpha = 0.020\,43$. In contrast with the previous example in figure 3, the initial wave profile shows five dominant waves per group length, and the cyclic period for change of envelope shape is almost twice that of figure 3. The cyclic period for the first example, in figure 2, is about 1.5 times that in figure 4. Unlike figure 2, the groups in figure 4 occur at regular intervals along an otherwise nearly uniform wavetrain, and they have a continuous identity as they propagate.

5. Discussion

The three examples illustrate how the nonlinear interactions between wave components concentrate wave energy locally in space and time, causing in some cases almost a doubling of wave heights. This mechanism provides one explanation for local wave breaking on the ocean surface, where the wave energy is concentrated sufficiently that the wave at the centre of a sharply peaked group disintegrates. The breaking property cannot be demonstrated by a spectral method, but is a logical extension of the properties illustrated. Three-dimensional effects are likely to be significant on the ocean surface, in addition to the two-dimensional cyclic concentration of wave energy described here.

An intriguing possibility worth investigation is that cyclically recurring wave groups may be part of a progression to chaotic motion on the ocean surface. A review by Miles (1984) summarizes the progression as consisting of fixed points, limit cycles, multiply periodic limit cycles, and chaotic motion on strange attractors. The fixed points here are the wave groups of constant shape, and the limit cycles are the wave groups of cyclically recurring shape. The next step would be to make the model more realistic by adding empirical relations for forcing and damping, both with wavenumber or frequency dependence. If the progression to chaotic motion is applicable, then for certain parameter ranges involving resonant forcing and weak damping, the wave motion would evolve to a continuous wave spectrum. Phillips (1985) identified wave forcing, nonlinear wave interactions, and wave breaking as being the dominant contributing factors to the equilibrium range in ocean wave spectra. It may be possible, when more powerful computing resources become available, to integrate numerically a resonantly forced, damped, unidirectional, fully nonlinear model of the ocean wave surface from an initial modulated wavetrain to an equilibrium wave spectrum such as that described by Phillips. However, as Newell (1986) has indicated, it is still a major problem to understand the transition from a spatially regular model with a chaotic temporal behaviour to a model of the ocean surface in which the temporal and spatial power spectra are both broadband.

Most of this investigation was done during a period of leave at the Institute for Geophysics and Planetary Physics, University of California, San Diego, as a guest of John W. Miles. Acknowledgement is made to the National Center for Atmospheric Research and to the San Diego Supercomputer Center, both sponsored by the National Science Foundation, for the computing time used in this research.

REFERENCES

BRYANT, P. J. 1982 Modulation by swell of waves and wave groups on the ocean. *J. Fluid Mech.* **114**, 443–466.

BRYANT, P. J. 1985 Doubly periodic progressive permanent waves in deep water. *J. Fluid Mech.* **161**, 27–42.

DOLD, J. W. & PEREGRINE, D. H. 1986 Water-wave modulation. *School of Mathematics, University of Bristol, Rep.* AM-86-03.

LAKE, B. M., YUEN, H. C., RUNGALDIER, H. & FERGUSON, W. E. 1977 Nonlinear deep-water waves: theory and experiment. Part 2. Evolution of a continuous wave train. *J. Fluid Mech.* **83**, 49–74.

MILES, J. 1984 Strange attractors in fluid dynamics. *Adv. Appl. Mech.* **24**, 189–214.

NEWELL, A. C. 1986 Chaos and turbulence: Is there a connection? In *Mathematics Applied to Fluid Mechanics and Stability: Proc. Conf. Dedicated to Richard C. DiPrima* (Ed. D. A. Drew & J. E. Flaherty), pp. 157–189. SIAM.

PHILLIPS, O. M. 1985 Spectral and statistical properties of the equilibrium range in wind-generated gravity waves. *J. Fluid Mech.* **156**, 505–531.

YUEN, H. C. & FERGUSON, W. E. 1978 Relationship between Benjamin–Feir instability and recurrence in the nonlinear Schrödinger equation. *Phys. Fluids* **21**, 1275–1278.